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ON THE FORM OF THE HIGH REYNOLDS NUMBER EXPANSION SOLUTION FOR INCOMPRESSIBLE FLOW OVER A FLAT PLATE

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For the past few years it has been rather generally believed that the asymptotic expansion for large Reynolds number of the Navier-Stokes solution for laminar viscous incompressible flow over a semi-infinite flat plate immersed in a uniform stream must contain terms involving the logarithm of Reynolds number. Goldstein[1,2] and Imai[3] have concluded that, in order to satisfy the condition of exponential decay of vorticity through the boundary layer, the solution, exhibited, for example, by the local skin-friction coefficient $C_f = \tau_0/(1/2)\rho U^2$, where τ_0 is the shear stress at the plate, must have the form (nonanalytic with respect to $R_x^{-1/2}$ at $R_x = \infty$):

$$C_f = a_1 R_x^{-1/2} + a_2 R_x^{-1} + a_{31} R_x^{-3/2} \ln R_x + a_{32} R_x^{-3/2} + o(R_x^{-3/2}) \quad \text{as} \quad R_x \to \infty \tag{1}$$

where R_x is the local Reynolds number, $U\bar{x}/\nu$. The "smaller order" symbol in the expression $A = o(B)$ as $\epsilon \to 0$ means $\lim_{\epsilon \to 0} (A/B) = 0$. The fact that their solutions are given only to within an undetermined constant to this order has not been satisfactorily explained, although it has been discussed by a number of authors (see Van Dyke[4]).

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A method of asymptotic expansions for solving singular perturbation problems has been developed[5] and may be considered to be a new approach to, and a modification and extension of, the method of inner and outer expansions developed by Kaplun, Lagerstrom, and Cole.[6-10] Application of the method in Ref. 5 to the flat-plate problem described above has led to the systematic determination of the expansion solution about $R_x = \infty$ which is completely determined to any order. The solution does not contain any logarithmic terms. It not only satisfies the Navier-Stokes equations and the usually imposed boundary conditions which define the problem, but also satisfies <u>automatically</u> the requirement of exponential decay of vorticity. Satisfaction of the latter condition was forced in the solutions of Goldstein[1,2] and Imai[3] by the otherwise arbitrary addition of the logarithmic term. The expansion solution given in Ref. 5 in terms of the "displacement variable" $R_{\bar{x}} = R_x + \beta^2$ ($\beta = 0.8603935$) is believed to be valid (convergent) arbitrarily close to the leading edge.

It is the purpose of this note to explain and clarify the reasons why the logarithmic terms should not be present in the expansion solution for large R_x (i.e., expansion about $R_x^{-1/2} = 0$).

First, it must be pointed out that, although several authors have attributed the "need" for log terms in, and thus a nonanalytic (with respect to $R_x^{-1/2}$) form of, the expansion solution for large R_x to the singularity at the leading edge of the plate, the singularity at $R_x = 0$ need not affect the form of the expansion about $R_x = \infty$ ($R_x^{-1/2} = 0$). Thus, even though C_f is nonanalytic with respect to R_x (or even $R_x^{1/2}$) at $R_x = 0$, if it is analytic with respect to $z \equiv R_x^{-1/2}$ at $z = 0$, it can be expanded in a Maclaurin series about $z = 0$ in the form

$$C_f = a_0 + a_1 R_x^{-1/2} + a_2 R_x^{-1} + a_3 R_x^{-3/2} + a_4 R_x^{-2} + \ldots \quad (2)$$

(where it is known from the Blasius solution to first-order boundary-layer theory that $a_0 = 0$ and $a_1 = 0.66411$).

Secondly, we consider now the possibility that the solution at $R_x = \infty$ is nonanalytic by virtue of a logarithmic singularity, so that the asymptotic expansion would contain log terms. In Ref. 5 the solution was assumed to be analytic with respect to $R_x^{-1/2}$ at $R_x = \infty$. The question then arises whether logarithmic terms required for a valid solution could have been omitted by assuming the analytic form of solution. For the present problem (which is included in a large class) they could not, as will be shown.

The dimensionless stream function ψ is defined by $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$ where u and v are the velocity components made dimensionless with respect to the free-stream velocity U, where x and y are the coordinates parallel and normal to the plate made dimensionless with respect to some arbitrary but fixed length L, and where $R = UL/\nu = R_x/x$. It is known that R can be eliminated completely from the problem by letting $\overline{\Psi} = R\psi$, $\overline{X} = Rx = R_x$, and $\overline{Y} = Ry$ (see Ref. 5, Eqs. (248), (249), and (252)). Hence, even though R may be separated from x and y in order to find the asymptotic expansion as $R \to \infty$, the final result for ψ, and all results obtainable from ψ, must be reducible to a form where ψ, x, y, and R occur only in their product forms such that $R\psi$ is a function of (Rx, Ry). For example, the local skin-friction coefficient is, by definition

$$C_f = (2/R)\psi_{yy}(x,o) = 2[\partial^2(R\psi)/\partial(Ry)^2]_{Ry=o} = \text{function of } Rx \tag{3}$$

Therefore R and x can occur in the solution only to the extent that they can be combined as a product. If C_f has an expansion in terms of the small quantity $R^{-1/2}$, such as

$$\begin{aligned} C_f &= R^{-1/2}b_1(x) + R^{-1}b_2(x) + (R^{-3/2}\,ln\,R)b_{31}(x) \\ &\quad + R^{-3/2}b_{32}(x) + \ldots \qquad \text{as } R^{-1/2} \to 0 \end{aligned} \tag{4}$$

and if the term of order $R^{-3/2}\,ln\,R$ as $R^{-1/2} \to 0$ actually exists in this expansion, then $b_{32}(x)$ must contain a term proportional to $x^{-3/2}\,ln\,x$, since

$$(Rx)^{-3/2}\,ln(Rx) = (R^{-3/2}\,ln\,R)x^{-3/2} + R^{-3/2}x^{-3/2}\,ln\,x \tag{5}$$

If $b_{32}(x)$ does not contain the term proportional to $x^{-3/2}\,ln\,x$, then there can be no term to the order $R^{-3/2}\,ln\,R$ as $R \to \infty$. Furthermore, if the expansion must, in fact, include one or more terms, each containing $ln\,R$ to some positive integer power m, then the need for including each such term will be indicated by the presence of the quantity $ln\,x$ to an appropriate power in the terms involving an appropriate power of $R^{-1/2}$ in Eq. (4). If the evaluated function of x multiplying $(R^{-1/2})^n$ $(n \geq 0)$ in Eq. (4) contains the term $x^{-(1/2)n}(\ln x)^m$, then, in general, there must be terms in the correct expansion which are constant multiples of each of the $m + 1$ terms on the right side of Eq. (6b):

$$(Rx)^{-(1/2)n}(\ln Rx)^m = R^{-(1/2)n}x^{-(1/2)n}(\ln R + \ln x)^m \tag{6a}$$

$$= R^{-(1/2)n}x^{-(1/2)n}[(\ln R)^m + m(\ln R)^{m-1} \ln x + (1/2!)m(m-1)(\ln R)^{m-2}(\ln x)^2 + \dots + (\ln x)^m] \tag{6b}$$

It should also be noted that, in a given problem for which an expansion does include log terms as well as power terms, the power terms are in no way affected by the inclusion or omission of the terms of logarithmic order in the assumed form of the expansion solution, since the powers and the logs cannot "mix" to give an order of magnitude which is purely a power of the small quantity as it approaches zero.

In Ref. 5 the solution is found for $R^{1/2}\psi(x,y;R) = \Psi(x,Y;R)$ as $R \to \infty$ holding Ψ, x, and $Y = R^{1/2}y$ fixed, by assuming an analytic (power series) expansion in terms of the small quantity $R^{-1/2}$. Self-similar solutions for the respective terms are obtained of which the validity cannot be doubted. The respective terms are not logarithmic with respect to x or Y and hence, according to the above arguments, there can be no terms of logarithmic order as R or $R_x \to \infty$. Evidently the analytic form of the expansion solution (including Eq. (2)) is valid, since the problem is completely satisfied by that form. In Ref. 5 the solution is also given in terms of the displacement variable $R_{\tilde{x}} = R_x + \beta^2$ in the form

$$C_f = \tilde{a}_1 R_{\tilde{x}}^{-1/2} + \tilde{a}_2 R_{\tilde{x}}^{-1} + \tilde{a}_3 R_{\tilde{x}}^{-3/2} + \dots \qquad (7)$$

(where $\tilde{a}_1 = 0.6641146$, $\tilde{a}_2 = 0$, $\tilde{a}_3 = 1.5695$, and $\tilde{a}_4 = 0$) which is equivalent to Eq. (2) with different values for the constants and which probably converges for all $R_{\tilde{x}} > \beta^2 (R_x > 0)$.

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